

# IUCAA GR Refresher Course Tutorials by BM

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## Tutorial:1 (23-06-2025)

### Problem:1 Derive Maxwell's Equations from the Action Principle

The electromagnetic field can be described by the Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - J^\mu A_\mu,$$

where  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$  is the electromagnetic field strength tensor, and  $J^\mu$  is the four-current.

Using the **action principle**, vary the action with respect to the four-potential  $A^\mu$  to derive the inhomogeneous **Maxwell's equations**:

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

#### Instructions:

- 1. Write down the action  $S = \int d^4x \mathcal{L}$ .
- 2. Perform a variation  $A^\mu \rightarrow A^\mu + \delta A^\mu$  and compute  $\delta S$ .
- 3. Use integration by parts and discard boundary terms to isolate the Euler-Lagrange equation.
- 4. Conclude with the field equation for  $A^\mu$  and interpret it as Maxwell's equations in covariant form.

### Problem:2 Derive the Continuity Equation from Maxwell's Equations

In Problem 1, you showed that the inhomogeneous Maxwell equations in covariant form can be written as:

$$\partial_\mu F^{\mu\nu} = J^\nu$$

where  $F^{\mu\nu}$  is the electromagnetic field strength tensor and  $J^\nu$  is the four-current density.

**Task:**

Using the above relation and the antisymmetry of the electromagnetic field tensor  $F^{\mu\nu} = -F^{\nu\mu}$ , prove that the four-current is conserved:

$$\partial_\mu J^\mu = 0$$

**\*\*Hints:\*\***

- Take the partial derivative  $\partial_\nu$  of both sides of the equation  $\partial_\mu F^{\mu\nu} = J^\nu$ .
- Use the fact that mixed partial derivatives commute:  $\partial_\nu \partial_\mu = \partial_\mu \partial_\nu$ .
- Use the antisymmetry of  $F^{\mu\nu}$  to simplify.

## Problem:3 Charge Conservation in Flat and Curved Spacetime

### (i) Charge Conservation in Flat Spacetime

The continuity equation in flat spacetime is given by:

$$\partial_\mu J^\mu = 0$$

where  $J^\mu = (\rho, \vec{J})$  is the four-current density. Show that the spatial integral of  $J^0(\vec{x}, t)$ , the charge density, is conserved in time:

$$\frac{d}{dt} \int_{R^3} J^0(\vec{x}, t) d^3x = 0$$

Hence, interpret this integral as the total conserved electric charge in the system.

### (ii) Charge Conservation in Curved Spacetime

Show that, in curved spacetime, the continuity equation generalizes to:

$$\nabla_\mu J^\mu = 0$$

where  $\nabla_\mu$  is the covariant derivative compatible with the metric  $g_{\mu\nu}$ . Find the total conserved electric charge in curved space-time from the continuity equation in curved space-time involving the covariant derivative.